ECCENTRICITY OF THE NORMAL ELLIPSOID

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The normal gravity field is a reference surface for the external gravity field of the earth. The source of the normal gravity field is a model earth which bets fits the actual shape of the earth. An ellipsoid (an ellipse of revolution) is assumed for the model earth and this ellipsoid is said to have the same mass M of the earth, but with homogenous density; the same angular velocity ω; and the surface of this ellipsoid is said to be a level surface (an equipotential surface) of its own gravity field. This geocentric equipotential ellipsoid is the basis of the **Geodetic Reference System 1980** (GRS80) and is defined by the following conventional constants (Moritz 2000)

- Equatorial radius of the earth

  $a = 6378137$ m

- Geocentric gravitational constant of the earth (including the atmosphere)

  $GM = 3.986005E+14\ \mathrm{m^3/s^2}$

- Dynamical form factor of the earth, excluding the permanent tidal deformation

  $J_2 = 1.08263E-03$

- Angular velocity of the earth

  $\omega = 7.292115E-05$ rad/s

The fundamental derived constant is the square of the first eccentricity e^2 and this quantity is linked to the defining constants via

$$e^2 = 3J_2 + \frac{4}{15}\frac{\omega^2 a^3}{GM}\frac{e^3}{2q_0} \tag{1}$$

Where

$$q_0 = \frac{1}{2}\left\{\left(1+\frac{3}{e'^2}\right)\tan^{-1} e' - \frac{3}{e'}\right\} \tag{2}$$

$$e' = \frac{e}{\sqrt{1-e^2}} \quad \text{(second eccentricity)} \tag{3}$$

Moritz (2000) outlines the derivation of (1) by reference to Heiskanen & Moritz (1967). A derivation of (1) is also given in Deakin (1997, pp. 34-35).

Other geometric constants of the normal ellipsoid can be computed by the formula

$$b = a\sqrt{1-e^2} = a(1-f) \quad \text{(semi-major axis)} \tag{4}$$

$$f = \frac{a-b}{a} = 1-\sqrt{1-e^2} \quad \text{(flattening)} \tag{5}$$

$$E = \sqrt{a^2-b^2} \quad \text{(linear eccentricity)} \tag{6}$$

Note also that the eccentricities and axes lengths are linked by

$$e = \frac{E}{a}, \quad e' = \frac{E}{b}, \quad \frac{e}{e'} = \frac{b}{a} \tag{7}$$

e^2 cannot be evaluated directly from (1) since it appears on both sides of the equals sign, but instead must be solved iteratively. As a preliminary to developing an iterative solution it is useful to first consider an alternative expression for q_0 given in (2).

The series for the trigonometric function $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$ for $|x| < 1$ and since $e' < 1$ we may write

$$q_0 = \frac{1}{2}\left\{\left(1+\frac{3}{e'^2}\right)\left(e' - \frac{1}{3}e'^3 + \frac{1}{5}e'^5 - \frac{1}{7}e'^7 + \cdots\right) - \frac{3}{e'}\right\}$$

and

$$\begin{aligned} q_0 &= \frac{1}{2}\left\{e' - \frac{1}{3}e'^3 + \frac{1}{5}e'^5 - \frac{1}{7}e'^7 + \frac{1}{9}e'^9 - \cdots + \frac{3}{e'} - e' + \frac{3}{5}e'^3 - \frac{3}{7}e'^5 + \frac{3}{9}e'^7 - \cdots - \frac{3}{e'}\right\} \\ &= \frac{2}{3\cdot 5}e'^3 - \frac{4}{5\cdot 7}e'^5 + \frac{6}{7\cdot 9}e'^7 - \frac{8}{9\cdot 11}e'^9 + \cdots \end{aligned} \tag{8}$$

giving (Heiskanen & Moritz 1967, p. 72)

$$q_0 = -\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)} e'^{2n+1} \tag{9}$$

Using (9) we may develop an expression for $\frac{2q_0}{e^3}$ [the reciprocal of the last term in (1)] as

$$\frac{2q_0}{e^3} = -\sum_{n=1}^{\infty}(-1)^n \frac{4n}{(2n+1)(2n+3)} \frac{e'^{2n+1}}{e^3} \tag{10}$$

Using (3) gives $e'^{2n+1} = e^{2n+1}\left(\frac{1}{1-e^2}\right)^{\frac{2n+1}{2}}$ and denoting $x_n = \frac{e'^{2n+1}}{e^3}$ and simplifying gives

$$x_n = \frac{e^{2n-2}}{1-e^2}\left(\frac{1}{1-e^2}\right)^{\frac{2n+1}{2}} \tag{11}$$

Using (11), equation (10) can be written as

$$\frac{2q_0}{e^3} = -\sum_{n=1}^{\infty}(-1)^n \frac{4n}{(2n+1)(2n+3)} x_n \tag{12}$$

where, for the sequence $n = 1, 2, 3, 4, \ldots$

$$x_1 = \frac{1}{1-e^2}\left(\frac{1}{1-e^2}\right)^{\frac{1}{2}} = \left(\frac{1}{1-e^2}\right)^{\frac{3}{2}}$$

$$x_2 = \frac{e^2}{1-e^2}\left(\frac{1}{1-e^2}\right)^{\frac{3}{2}} = \frac{e^2}{1-e^2}x_1 = e'^2 x_1$$

$$x_3 = \frac{e^4}{1-e^2}\left(\frac{1}{1-e^2}\right)^{\frac{5}{2}} = \frac{e^2}{1-e^2}\left(\frac{e^2}{1-e^2}\right)\left(\frac{1}{1-e^2}\right)^{\frac{3}{2}} = \frac{e^2}{1-e^2}x_2 == e'^2 x_2$$

$$x_4 = \frac{e^6}{1-e^2}\left(\frac{1}{1-e^2}\right)^{\frac{7}{2}} = \frac{e^2}{1-e^2}\left(\frac{e^4}{1-e^2}\right)\left(\frac{1}{1-e^2}\right)^{\frac{5}{2}} = \frac{e^2}{1-e^2}x_3 == e'^2 x_3$$

In any iterative process an initial or starting vale must be known (or assumed). A starting value for e^2 may be obtained considering the following:

1. Using (8) an alternative expression for $2q_0$ can be obtained as

$$2q_0 = \frac{4}{15} e'^3 \left(1 - \frac{6}{7} e'^2 + \frac{5}{7} e'^4 - \frac{20}{33} e'^6 + \cdots \right) \tag{13}$$

2. Dividing (13) by e^3 and re-arranging gives

$$\frac{15}{4} \frac{2q_0}{e^3} = \frac{e'^3}{e^3} \left(1 - \frac{6}{7} e'^2 + \frac{5}{7} e'^4 - \frac{20}{33} e'^6 + \cdots \right) \tag{14}$$

3. Using (7) and (3) in (14) gives

$$\frac{15}{4} \frac{2q_0}{e^3} = \left(\frac{1}{\sqrt{1-e^2}}\right)^3 \left(1 - \frac{6}{7}\left(\frac{e^2}{1-e^2}\right) + \frac{5}{7}\left(\frac{e^2}{1-e^2}\right)^2 - \frac{20}{33}\left(\frac{e^2}{1-e^2}\right)^3 + \cdots \right) \tag{15}$$

4. Expanding the right-hand-side of (15) into a power series in e^2 gives

$$\frac{15}{4} \frac{2q_0}{e^3} = 1 + \frac{9}{14} e^2 + \frac{25}{56} e^4 + \frac{175}{528} e^6 - \frac{375}{1408} e^8 - \cdots \tag{16}$$

5. Inverting the series on the right-hand-side of (16) gives

$$\frac{4}{15} \frac{e^3}{2q_0} = 1 - \frac{9}{14} e^2 - \frac{13}{392} e^4 - \frac{4189}{181104} e^6 + \frac{1720993}{3380608} e^8 + \cdots \tag{17}$$

So, from (17) $\frac{4}{15} \frac{e^3}{2q_0} \approx 1$ and substituting into (1) gives the starting value of e^2 as

$$e^2_{START} = 3J_2 + \frac{\omega^2 a^3}{GM} \tag{18}$$

Maxima[1] code for the evaluation of e^2 is shown below, and this algorithm is based on the FORTRAN subroutine *REFVAL* given in Tscherning, Rapp & Goad (1983)

Notes on minor corrections to original version.

1. Author location changed from Bonbeach, VIC, 3196 to Dunsborough, WA, 6281
2. Last four digits of the number string for 1/f (see overleaf) changed from 4820 to 5753. I was alerted to this error by John Nolton (email of 02-Jul-2019) and I thank him for his diligent work. This error was corrected in the Maxima code (see overleaf) by increasing the maximum number of iterations in the for-loops from 20 to 40.

[1] Maxima is a computer algebra system that yields high precision numerical results by using exact fractions, arbitrary precision integers, and variable precision floating point numbers. http://maxima.sourceforge.net/

```
/*********************************************************************/
/* refval.mac                                                        */
/* Maxima program for calculation of eccentricity-squared and        */
/* flattening of the Normal ellipsoid defined by the four defining   */
/* constants:                                                        */
/*     a = 6378137 m (equatorial radius of the earth),               */
/*    GM = 3.986005e+14 m^3/s^2 (geocentric gravitational constant), */
/*    J2 = 1.08263e-03 (dynamical form factor),                      */
/* omega = 7.292115e-05 rad/s (angular velocity of earth)            */
/*********************************************************************/

/* set precision for bigfloat variables*/
fpprec:60$

/* set values for the defining constants a, GM, J2 and omega */
a     : 6378137.0b0$
GM    : 3.986005b14$
J2    : 1.08263b-3$
omega : 7.292115b-5$

/* set starting value of e2 */
m1 : omega^2*a^3/GM$
e21 : 3*J2 + m1$
/* set e2 = initial value */
e2 : e21$
for k:1 thru 40 step 1 do block
  (ep2 : e2/(1-e2),
   x : (1/(1-e2))^(3/2),
   sgn : 1,
   twoq0 : 0,
   for j:1 thru 40 step 1 do block
     (twoq0 : twoq0 + (sgn*(4*j/(2*j+1)/(2*j+3)*x)),
      sgn : -sgn,
      x : x*ep2),
   e2 : e21 + (m1*((4/15/twoq0)-1)))$

/* e2 is now known.  The flattening f and flat = 1/f can be computed */
f : 1-sqrt(1-e2)$
flat : 1/f$

printf(true,"~1%~a~49,45h","  e2 = ",e2)$
printf(true,"~1%~a~49,45h","   f = ",f)$
printf(true,"~1%~a~49,45h"," 1/f = ",flat)$
```

Output from Maxima program *refval.mac*

Eccentricity-squared e^2, flattening f and reciprocal of flattening $1/f$ (45 decimal digits)

```
  e2 =   0.006694380022903415749574948586289306212443890
   f =   0.003352810681183637418165046184764464865509509
 1/f = 298.257222100882711243162836607614495018656495753
```

REFERENCES

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Heiskanen, W.E. and Moritz, H., 1967, *Physical Geodesy,* W.H. Freeman & Co., San Francisco, 364 pages.

Moritz, H., 2000, 'Geodetic Reference System 1980', *Journal of Geodesy,* Vol. 74, Issue 1, March 2000, pp. 128-133.

Tscherning, C.C., Rapp, R.H. and Goad, C., 1983, 'A comparison of methods for computing gravimetric quantities from high degree spherical harmonic expansions', *Manuscripta Geodaetica,* Vol. 8, 1983, pp. 249-272.